Exploring Ellipses

Kepler's first law states that the orbits of the planets are ellipses with the sun at one focus. But what makes an oval an ellipse?

Drawing ellipses:
The definition of ellipse is an oval with the property that the sum of the distance from one point on the ellipse to one focus, plus the distance from that same point on the ellipse to the other focus, is a constant. This allows us a way to draw an ellipse with a pencil and piece of string.

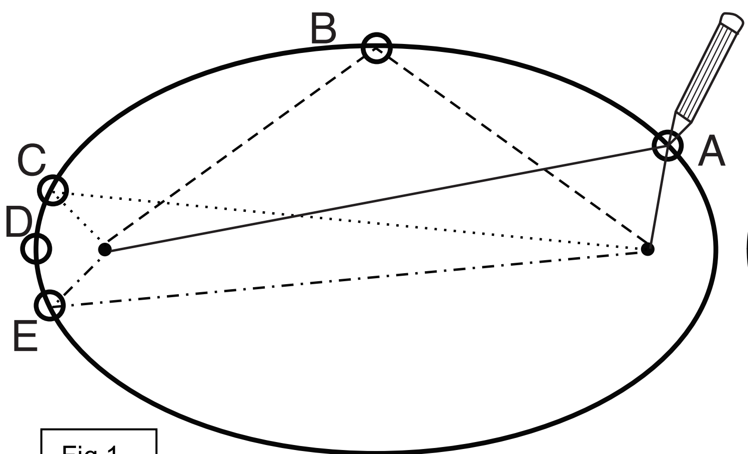


Fig 1

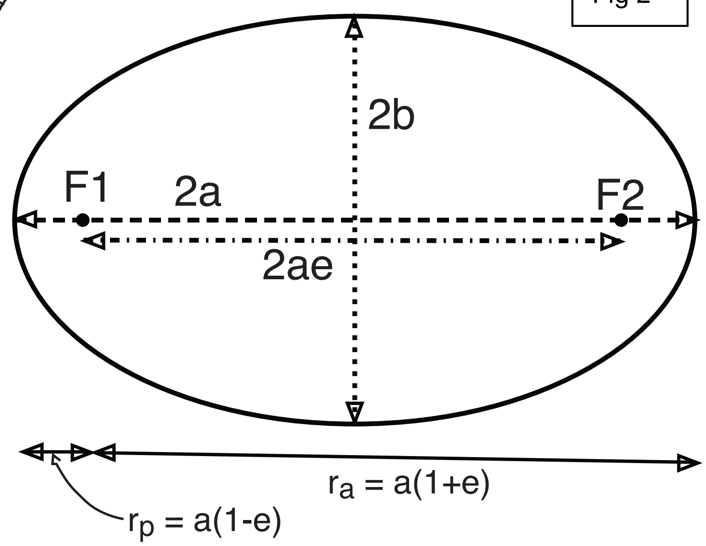


Fig 2

Take a length of string and tie each end to a pencil (if drawing it on a blackboard) or push-pin (if drawing it on a pad of paper). Use the entire length of the string to set up a horizontal line across the paper or blackboard – this is the long axis or "*major axis*" Have one student use the holders to fix the two ends of the string at two places along that long line, becoming the two focal points ("*foci*") (Fig 1). While that person holds tight, have a second student hold a drawing instrument in the loop of string, pulling the string tight as he or she draws the ellipse from A to B to C. Note at point "D" they need to lift up the string over the left-hand focus, or the string will wrap up. Have different students create different ellipses with the same length of string but with different distances between the two foci. How far apart do the foci need to be before the ellipse looks more like an oval and not a circle?

Since the long axis of the ellipse is called the "major axis", half that distance is called the "semi-major axis", often written as the letter **a**. The length of string used is equal to the distance 2a. The spacing between the two foci gives the *eccentricity* of ellipse. The definition of "*eccentricity,*" **e**, is the distance between the two foci, divided by the major axis; therefore, the distance between the two foci is 2ae (Fig 2). An eccentric ellipse is long and skinny, and has e near 1. What does an ellipse look like with e = 0?

Apogee and Perigee:
The maximum distance of a planet (or comet) from the sun is its *aphelion* distance, generally written as r_a. We say that the planet is "at aphelion". The minimum distance of a planet (or comet) from the sun is called is *perihelion*, and is written as r_p. When talking about satellites of the earth, we use the terms *apogee* for greatest distance and *perigee* for minimum distance. In each case the "apo" means largest and "peri"

smallest, and the suffix "helion" referers to the Sun and "gee" to the Earth. The terms in general are called "apoapsis" and "periapsis".

Looking at the graph, we see that the minimum distance r_p is equal to (a – ae), and the maximum distance r_a is equal to (a + ae). So another definition of eccentricity is the maximum distance minus minimum distance, divided by the average distance (which is a).

Your challenge:
Calculate the minimum and maximum distance from the Sun for each of the planets. Get the average distance a and eccentricity e from the Space Weather software, or from a textbook. Calculate the perihelion distance r_p and aphelion r_p for each. (1 AU is the distance from the Earth to the Sun). Which planet has the largest eccentricity?___________ The smallest? __________. Look at Pluto's perihelion distance – how does it compare to Neptune's aphelion distance?____________________

Planet	**a (AU)**	**e**	$\mathbf{r_p}$ = a - ae	$\mathbf{r_a}$ = a + ae
Mercury				
Venus				
Earth				
Mars				
Jupiter				
Saturn				
Uranus				
Neptune				
Pluto				

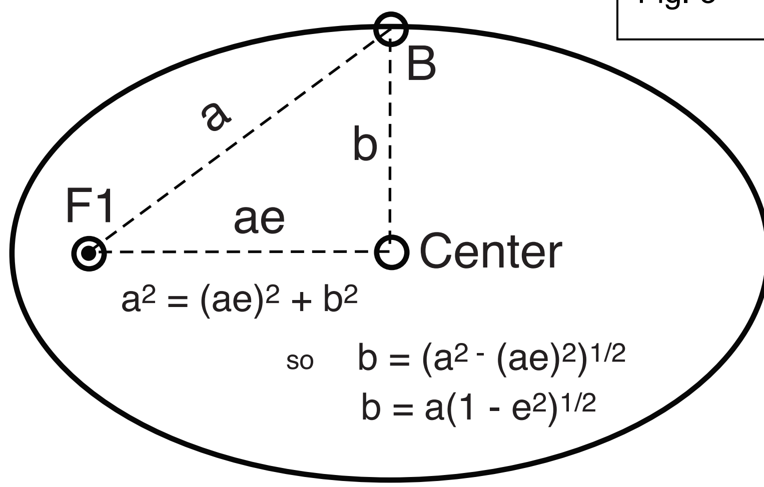
Going the other way:
If you know any two of the following: a, e, r_p or r_p, you can calculate the other two. For example, a comet may have a perihelion of 0.4 AU and an eccentricity of 0.9. What is its aphelion distance? (Hint: first calculate its semimajor axis, a ________ and then calculate the aphelion__________).

How Circular is that ellipse?
Page 4 shows a sequence of ellipses, with the eccentricities known but not shown on the graph. Have the students guess where the foci should be in each case. (You should have the students, in pairs, each trace two of the ellipses and place their guess as the locations of the foci on the line segment to the right).

Note that the top few ellipses look very very circular; however, they are not.. their major axes are just a little larger than their minor axes, but the foci are definitely offset. In fact, the eccentricities are 0 (to left), 0.1 (top right), 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95 (bottom left), and 0.98 (bottom right). Have the student trace the ellipse that is closest to the Earth's orbit and to Pluto's orbit. It's not surprising that Ptolemy could approximate the orbits of the planets very well by just using an offset circle!

Common misconception:
Students may think that the eccentricity is defined by the length of the short axis (2b on Figure 2) divided by the length of the major axis 2a. It is not. And in fact, students who know a right triangle and the Pythagorean theorem can easily calculate the distance b. Look at Figure 3. The distance from the focus F1 to point B must be equal to half the string length, or a. (Remember how we drew point B?). The distance from B to the Center is b (the semi-minor axis), and the distance from the Center to the focus F1 is ae. This is a right triangle, since BC is perpendicular to F1C. From the Pythagorean theorem $(a)^2 = (ae)^2 + b^2$, then $b^2 = (a)^2 - (ae)^2$, so $b^2 = a^2 (1 - e^2)$, and $b = a \sqrt{(1 - e^2)}$..

Fig. 3

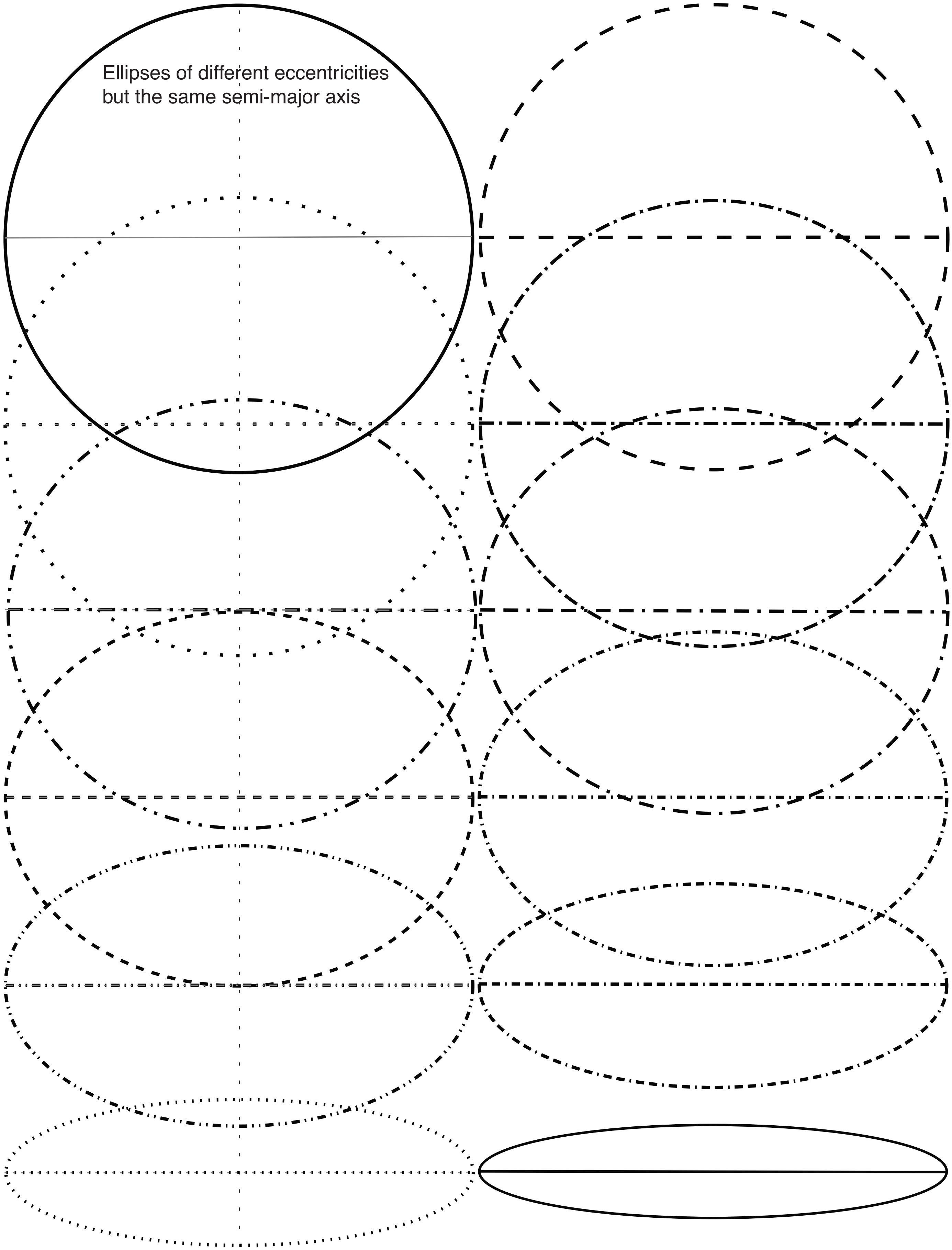


Advanced activities:
Measure the major and minor axes of the ellipses on the next page, and calculate their eccentricities using the formulas above. (Have each student choose two or three to do – work in pairs if appropriate). What is your estimate of the error of your measurement? (If your printer cuts off the edges of that page, you should still be able to measure one half of the minor axis, b, and the semimajor axis, a, for each ellipse.)

Recall that since $(a)^2 = (ae)^2 + b^2$, can you solve for e? [answer: $e = \text{sqrt}((a^2 - b^2) / a^2)$]
Use that formula to calculate e and ae for each ellipse, and plot the location of the two foci on the major axes.

On a piece of graph paper or graphing calculator, plot b as a function of e. Or, plot b^2 as a function of e. Can you extend the graph (extrapolate) to estimate b^2 when e gets larger than 1? Is that result physical (does it make physical sense?)

Ellipse number	**a**	**b**	**$a^2 - b^2$**	**e**


Ellipses of different eccentricities
but the same semi-major axis